Functional Analysis HW 6

Deadline: 3 Apr 2017

1. Let X be a normed space. For a subspace M of X and a subspace N of the dual space X^*. The annihilator of M is defined by

$$M^\perp := \{x^* \in X^* : x^*(m) = 0 \ \forall m \in M\} \quad \text{and} \quad {}^\perp N := \{x \in X : x^*(x) = 0 \ \forall \in x^* \in N\}.$$

Show that

(i) ${}^\perp(M^\perp) = \overline{M}$.

(ii) If $T : X \to Y$ is a bounded linear operator, then $\ker T^* = (im(T))^\perp$ and $\ker T =^\perp (imT^*)$.

2. Let $T : X \to Y$ be a bounded linear operator between the Banach spaces X and Y. Suppose that the quotient space Y/imT (as a vector space only) has finite dimension and thus, there is a finite dimensional vector subspace Z of Y such that $Y = imT \oplus Z$ (as the direct sum of vector spaces). Define a linear operator $S : (X/kerT) \oplus_\infty Z \to Y$ by

$$S : (\bar{x}, z) \in (X/\ker T) \oplus_\infty Z \mapsto Tx + z \in Y$$

where $(X/\ker T) \oplus_\infty Z$ denotes the direct sum $(X/\ker T) \oplus Z$ when it is endowed with the sup-norm $\|\cdot\|_\infty$. Show that

(i) S is a linear homeomorphism (Hint: by using the Open Mapping Theorem).

(ii) imT is a closed subspace. (Hint: the quotient space $X/\ker T$ can be viewed as a closed subspace of $(X/\ker T) \oplus_\infty Z$ under the natural isometric embedding $i : \bar{x} \in (X/kerT) \mapsto (\bar{x}, 0) \in (X/\ker T) \oplus_\infty Z$.)